TRINITY COLLEGE

Semester Two Examination, 2021

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1&2

Section One: Calculator-free

SOLUTIONS

WA student number: In figures | | | | | | | | |

In words ______________________________

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Your name ______________________________

Number of additional answer booklets used (if applicable): [ ]

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	51	35
Section Two: Calculator-assumed	13	13	100	92	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (51 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(4 marks)**

Let matrix $\boldsymbol{A} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$ and matrix $\boldsymbol{B} = \begin{bmatrix} 2k-1 & 1 \\ 2 & k+1 \end{bmatrix}$, where k is a constant.

(a) When $k = -1$, determine $\boldsymbol{AB}$. (2 marks)

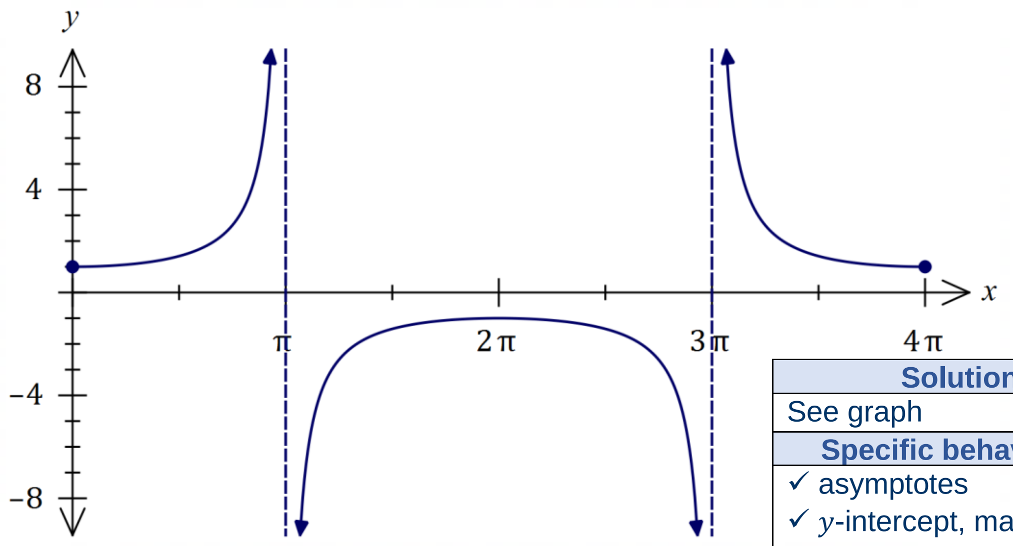
Solution
$\boldsymbol{AB} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix} \times \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix}$ $= \begin{bmatrix} 6 & -2 \\ -7 & 3 \end{bmatrix}$
Specific behaviours
✓ correct substitution   ✓ correct product

(b) Determine the value(s) of k if matrix $\boldsymbol{B}$ is singular. (2 marks)

Solution
$\det \boldsymbol{B} = 0$ $(2k-1)(k+1) - 2 = 0$ $2k^2 + k - 3 = 0$ $(2k+3)(k-1) = 0$ $k = 1, \quad k = -\frac{3}{2}$
Specific behaviours
✓ equates expression for determinant to zero   ✓ correct values of k

Question 2 **(6 marks)**

(a) Sketch the graph of $y = \sec\left(\frac{x}{2}\right)$ on the axes below for $0 \le x \le 4\pi$. (3 marks)



Solution
See graph
Specific behaviours
✓ asymptotes ✓ y-intercept, max ✓ correct curvature

(b) Prove the identity $\operatorname{cosec} 2A - \cot 2A = \tan A$. (3 marks)

Solution
$$\begin{aligned} LHS &= \operatorname{cosec} 2A - \cot 2A \\ &= \frac{1}{\sin 2A} - \frac{\cos 2A}{\sin 2A} \\ &= \frac{1 - \cos 2A}{\sin 2A} \\ &= \frac{2\sin^2 A}{2\sin A\cos A} \\ &= \frac{\sin A}{\cos A} \\ &= \tan A \\ &= RHS \end{aligned}$$
Specific behaviours
✓ converts between reciprocal and basic trig functions ✓ writes as single fraction ✓ uses double angle identities and simplifies

Question 3 **(6 marks)**

Let $z_1 = \sqrt{5} + 3i$ and $z_2 = \sqrt{5} - i$. Determine each of the following in the form $a + bi$.

(a) $i\bar{z}_1$. (2 marks)

Solution
$i(\sqrt{5} - 3i) = 3 + \sqrt{5}i$
Specific behaviours
✓ correct conjugate ✓ correct result

(b) $z_1 \times z_2$. (2 marks)

Solution
$(\sqrt{5} + 3i)(\sqrt{5} - i) = 5 + 3\sqrt{5}i - \sqrt{5}i + 3$ $= 8 + 2\sqrt{5}i$
Specific behaviours
✓ correctly expands ✓ correct result

(c) $z_1 \div z_2$. (2 marks)

Solution
$\frac{\sqrt{5} + 3i}{\sqrt{5} - i} \times \frac{\sqrt{5} + i}{\sqrt{5} + i} = \frac{2 + 4\sqrt{5}i}{6}$ $= \frac{1}{3} + \frac{2\sqrt{5}}{3}i$
Specific behaviours
✓ uses conjugate correctly ✓ correct result

Question 4 **(6 marks)**

(a) Determine the value(s) of the constant t given that $\begin{bmatrix} -2 & 4 \\ t & 8 \end{bmatrix}\begin{bmatrix} t \\ t \end{bmatrix} = \begin{bmatrix} 2t \\ 33 \end{bmatrix}$. (2 marks)

Solution
$t^2 + 8t = 33$ $t^2 + 8t - 33 = 0$ $(t + 11)(t - 3) = 0$ $t = -11, \quad t = 3$
Specific behaviours
✓ forms quadratic ✓ correct values

(b) Determine $\boldsymbol{A}^{-1}$ when $\boldsymbol{A} = \begin{bmatrix} 7 & 3 \\ -2 & 2 \end{bmatrix}$. (2 marks)

Solution
$\begin{bmatrix} 7 & 3 \\ -2 & 2 \end{bmatrix}^{-1} = \frac{1}{20}\begin{bmatrix} 2 & -3 \\ 2 & 7 \end{bmatrix}$
Specific behaviours
✓ indicates determinant ✓ correct inverse

(c) Show use of matrix methods to solve the following system of linear equations:

$$7x + 3y - 25 = 0$$
$$2y - 2x + 10 = 0$$

(2 marks)

Solution
$\begin{bmatrix} 7 & 3 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 25 \\ -10 \end{bmatrix}$ $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{20}\begin{bmatrix} 2 & -3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 25 \\ -10 \end{bmatrix}$ $= \begin{bmatrix} 4 \\ -1 \end{bmatrix}$
Specific behaviours
✓ writes as matrix equation ✓ shows use of inverse and obtains correct solution

Question 5 **(8 marks)**

(a) (i) Evaluate $2\sin\frac{5\pi}{12}\sin\frac{\pi}{12}$. (2 marks)

Solution
$$2\sin\frac{5\pi}{12}\sin\frac{\pi}{12}=\cos\left(\frac{4\pi}{12}\right)-\cos\left(\frac{6\pi}{12}\right)$$ $$=\frac{1}{2}$$
Specific behaviours
✓ uses correct product to sum identity ✓ evaluates

(ii) Hence evaluate $2\sin\frac{5\pi}{12}\sin\frac{3\pi}{12}\sin\frac{\pi}{12}$. (2 marks)

Solution
$$2\sin\left(\frac{5\pi}{12}\right)\sin\left(\frac{3\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)=2\sin\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)\sin\left(\frac{3\pi}{12}\right)$$ $$=\left(\frac{1}{2}\right)\sin\left(\frac{3\pi}{12}\right)=\left(\frac{1}{2}\right)\times\frac{1}{\sqrt{2}}$$ $$=\frac{1}{2\sqrt{2}}$$
Specific behaviours
✓ forms product ✓ evaluates

(b) Solve the equation $2\sin^2 2x = 3\cos 2x$, $0 \le x \le 2\pi$. (4 marks)

Solution
$$2(1-\cos^2 2x)-3\cos 2x=0$$ $$2\cos^2 2x+3\cos 2x-2=0$$ $$(2\cos 2x-1)(\cos 2x+2)=0$$ $$\cos 2x=\frac{1}{2}$$ $$2x=\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{11\pi}{3}$$ $$x=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$$
Specific behaviours
✓ uses Pythagorean identity to form quadratic ✓ factors quadratic and indicates one solution ✓ all correct solutions

Question 6 **(7 marks)**

(a) Determine all complex solutions to the equation $z^2 - 10z + 27 = 0$. (2 marks)

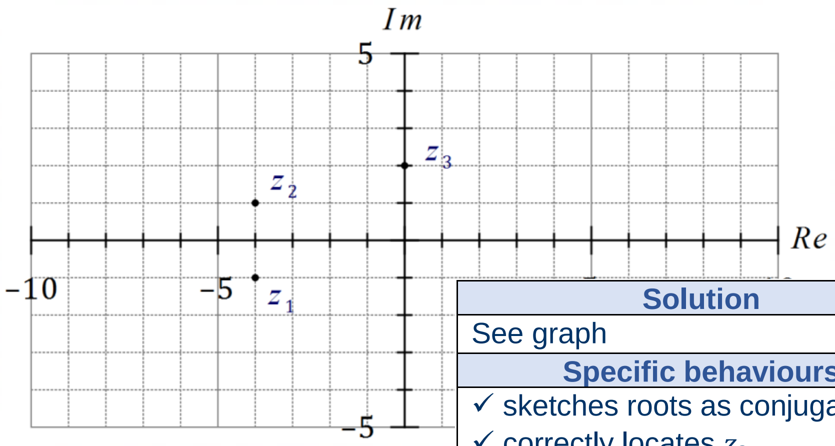
Solution
$(z-5)^2 = 25 - 27$ $(z-5)^2 = 2i^2$ $z = 5 \pm \sqrt{2}i$
Specific behaviours
✓ completes square ✓ both correct solutions

(b) $z_1 = -4 - i$ is a solution to $f(z) = 0$, where $f(z)$ is a real quadratic polynomial.

(i) State z_2, another solution to $f(z) = 0$. (1 mark)

Solution
$z = -4 + i$
Specific behaviours
✓ correct complex number

(ii) Let $z_3 = z_2 - z_1$. Plot and label z_1, z_2 and z_3 in the complex plane below. (2 marks)



Solution
See graph
Specific behaviours
✓ sketches roots as conjugate pair ✓ correctly locates z_3

(iii) Determine $f(z)$, given that the coefficient of its z^2 term is 1. (2 marks)

Solution
Let $f(z) = z^2 + bz + c$. Then $b = -2(-4) = 8$ and $c = 4^2 + 1^2 = 17$. $f(z) = z^2 + 8z + 17$
Specific behaviours
✓ shows sum and product of roots (or product of factors) ✓ correct equation

Question 7 **(6 marks)**

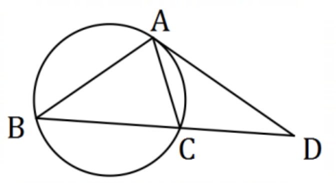
Use mathematical induction to prove that $2^{5n} - 5^n$ is divisible by 9 for all integers $n \geq 1$.

Solution
Let $f(n) = 2^{5n} - 5^n$. When $n = 1$ then $f(1) = 32 - 5$ $= 27$ $= 3 \times 9$ which is divisible by 9 Assume true for $n = k$ so that $2^{5k} - 5^k = 9I$ for some integer I. When $n = k + 1$ then $f(k+1) = 2^{5k+5} - 5^{k+1}$ $= 2^5 \cdot 2^{5k} - 5 \cdot 5^k$ $= 32(9I + 5^k) - 5 \cdot 5^k$ (using assumption) $= 32(9I) + 27(5^k)$ $= 9\left(32I + 3(5^k)\right)$ which is divisible by 9 Hence $f(n)$ is divisible by 9 for $n = k + 1$ and as demonstrated divisible for $n = 1$ then will be divisible for all $n \geq 1$.
Specific behaviours
✓ demonstrates true for $n = 1$ ✓ makes assumption for $n = k$ ✓ expression for $f(k+1)$ ✓ uses assumption to replace 2^{5k} ✓ factors out 9 from $f(k+1)$ ✓ concluding statement

Question 8 **(8 marks)**

(a) Points A, B and C lie on a circle.

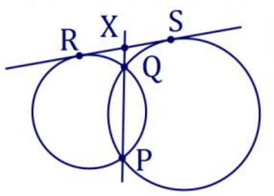
The tangent to the circle at A intersects secant BC at point D.



Prove that $AD^2 = BD \times CD$. (4 marks)

Solution
First prove that $\Delta ADC \sim \Delta BDA$: $\angle ADC = \angle BDA$ (common) $\angle DAC = \angle DBA$ (angles in opposite segments) Hence $\Delta ADC \sim \Delta BDA$ as two pairs of congruent angles. Using ratios of corresponding sides, $\frac{AD}{BD} = \frac{CD}{AD} \Rightarrow AD^2 = BD \times CD$.
Specific behaviours
✓ shows congruency of one pair of angles, with reasoning ✓ shows congruency of second pair of angles, with reasoning ✓ establishes similarity, with reasoning ✓ completes proof using ratio of sides

(b) Two unequal circles intersect at P and Q. A common tangent touches one circle at R and the other circle at S. PQ produced intersects RS at X. Prove that X bisects RS. (4 marks)

Solution
 Let PQ and RS intersect at X. In smaller circle, $RX^2 = XQ \times XP$ (tangent secant theorem) In larger circle, $SX^2 = XQ \times XP$ (tangent secant theorem) Hence $RX^2 = SX^2 \Rightarrow RX = SX$ and so PQ produced bisects RS.
Specific behaviours
✓ labelled diagram ✓ uses tangent-secant theorem in one circle ✓ uses tangent-secant theorem in the other circle ✓ concludes proof

End of questions

Supplementary page

Question number: _________